Nonlinear Diffusions and Entropy Dissipation: From Geometry to Biology

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1 Overview of the Field

Entropy functionals can be defined as Lyapunov functionals having special scaling properties or comparing well with other functionals, such as energy, that have a natural interpretation in some evolution equation. One natural source of such evolution equations is *gradient flow*. If an evolution equation can be written in the form of gradient flow for some functional, then formally at least, that functional will be monotone decreasing along the evolution.

It turns out that many dissipative equations can be written in gradient flow form. In fact, many such equations can be written in gradient flow form in many ways, since there can be compensation between a change in the functional and a change in the metric with respect to which the gradient of the functional is taken. For instance, the heat equation

$$\frac{\partial}{\partial t}\rho = \Delta\rho$$

is gradient flow for the energy integral $\|\nabla\rho\|_2^2$ in the L^2 metric, and also for the entropy functional $\int \rho \log \rho \mathrm{d}x$ with respect to the 2-Wasserstein metric. Already at the time of the previous B.I.R.S. meeting on this subject, 15-20 April 2006, the fundamental ideas originating with Otto on gradient flows in Wasserstein metrics had seen extensive development. Any evolution equation for densities ρ on $\mathbb{R}^n$ that is of the form

$$\frac{\partial}{\partial t}\rho = \mathrm{div}\left(\rho\,\mathrm{grad}\frac{\delta\mathcal{F}[\rho]}{\delta\rho}\right) , \tag{1}$$

for some functional $\mathcal{F}[\rho]$ is, formally at least, gradient flow for the 2-Wasserstein metric. There is now a well established methodology for studying such PDEs when the functional $\mathcal{F}$ is uniformly strictly displacement convex; i.e., convex with respect to the 2-Wasserstein metric. This methodology is particularly well-developed for investigating the large time behavior of solutions, where in many cases it yields an optimal rate estimate for convergence to equilibrium. Through this methodology, Wasserstein distances provide a common framework for continuous solutions of PDEs, their discrete analogs or intermediate measure-valued solutions.

However, Wasserstein distances, nor even gradient flows, are not all there is to the subject. Many very interesting dissipative evolutions have entropy functionals but cannot be written in gradient flow form in any useful way. The Boltzmann equation is one example, and at this meeting a number of other examples, especially coming from biology, were investigated. Likewise, though long-time behavior has been the focus of much work in the area, there has been very interesting progress in other directions. For example, quite literally in the "other direction" one can ask about "eternal solutions" to dissipative evolutions with steady states (such as the Boltzmann equation) or "ancient solutions" of dissipative evolution with a finite lifetime (such as Ricci flow).

The unification of the "entropy strategy" framework has recently enabled the researchers to recognize the common features in variety of systems and has accelerated the application to other sciences, including exchange of ideas between seemingly unrelated fields. Modern entropy based approaches have proven fruitful in studies of flows in porous media, population dynamics (chemotaxis and aggregation, structured population models, adaptive dynamics), thin liquid films, diffusion mediated transport in cells, reaction-diffusion systems, coagulation-fragmentation equations, and others.

As was evident at this meeting four years after the first one, and as we describe in the next section, recent advances and on-going work has expanded the scope of applicability of entropy methods in exciting ways.

2 Recent Developments, Open Problems, and Presentation Highlights

While there is a by now standard methodology for dealing with equations of the form (1) when $\mathcal{F}$ is uniformly strictly displacement convex, this is not the case when $\mathcal{F}$ fails to be displacement convex at all. Recent advances have permitted the treatment of at least some of such problem. Joint work of Matthes, McCann and Savare, and the talk of Blanchet, on joint work with Carlen and Carrillo, provide examples, in the first case for fourth order equations, and in the second case on the critical mass Keller-Segel model. In these works, the evolutions have a *second* Lyapunov functional, in addition to the primary one that drives the gradient flow, and this second Lyapunov function is displacement convex. A methodology for exploiting such such displacement convex secondary Lyapunov functionals is emerging. It is an open problem to develop a better understanding of why they occur, and how to find them, though the work of Matthes, McCann and Savare sheds some light on this.

Another very interesting problem is to consider what happens when (1) is changed to

$$\frac{\partial}{\partial t}\rho = \mathrm{div}\left(m(\rho)\ \mathrm{grad}\frac{\delta\mathcal{F}[\rho]}{\delta\rho}\right) \ , \tag{2}$$

for some nonnegative function m on $\mathbb{R}$ called the *mobility*. Many well-known evolution equations can be written in the form (2), but not (1). An important example is the Cahn-Hilliard equation, in which mobilities of the from $\rho(1-\rho) = \rho - \rho^2$ are natural. (Here, ρ is a density for one component of a binary alloy, and then $1-\rho$ is the density for the other. When $\rho(x) = 0$ or $\rho(x) = 1$, one has a pure state at x, and there is zero mobility.) One possible approach to this problem is to develop methods for working with metrics for which (2) describes the gradient flow of $\mathcal{F}$. Recent progress along these lines has been made by Carrillo, Dolbeault, Lisini, Matthes, Nazaret, Savaré, Slepčev, and Westdickenberg, working in various combinations; see the talk of Matthes. At least when the mobility is concave, and satisfies some growth conditions, interesting results can be obtained. However, there are many question in this area that remain open, and the meeting has made clear that we can expect exciting new developments here in the coming years.

The full scope of what one can do with Wasserstein like metrics in PDEs is only now becoming evident. When one is solving a gradient flow problem on a bounded domain, the variational problem associated to the gradient flow determines natural boundary condition. What can one do if one wants to study the same equation, but with some other boundary conditions? A natural guess is that one has to modify the metric or the functional, or both, so that the new boundary conditions become the natural boundary condition. Examples where this has been done by modifying the functional only go back to Otto's work on the thin film equation with fixed contact angle boundary conditions. Other problems seem to require a change in metric, and here progress is much more recent. One successfully treated example is the subject of Figalli's talk.

The talk of Westdickenberg, on joint work with Gangbo, presents an entirely different direction in the use of Wasserstein metrics in PDEs. These authors develop a second-order differential calculus for the Wasserstein metric and apply it to the isentropic Euler flow.

Another full line of research that has taken lot of attention in the last few years and during the conference are issues related to the aggregation equation. These are equations of the form (1) with the entropy functional

$$\mathcal{F}[\rho] = \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} W(x-y)\rho(x)\rho(y)\,dx\,dy\,.$$

Here, the potential W models the interaction between particles which in many applications in biology is a radial smooth potential with a singularity at the origin. Global unique measure solutions using the gradient-flow structure of the equation were obtained in a recent work by Carrillo, DiFrancesco, Figalli, Laurent, and Slepčev. On the other hand, for attractive potentials singular at the origin, finite time or infinite time blow-up can happen for solutions in the space L^1-L^p with p depending on the singularity of the potential as recently reported in works of Bertozzi, Carrillo, Laurent, and Rosado. Laurent's talk was discussing cases of the attractive singular potential for which there is finite time blow-up and how the solution behaves after that. Fellner's talk was treating the set of stationary solutions with long-range attractive - short-range repulsive potentials showing that this set can have a very complicated structure and how this set depends on the repulsive singularity at the origin.

Connecting with the previous item, a natural question arises. What happens to the aggregation equations when diffusion is added? There are many interesting question in this direction that will open new directions in the years to come. A result to mentioned in this direction was given at the workshop by Calvez. In his talk, he was showing that taking homogeneous interaction potentials $W(x) = |x|^k/k$ and linear/nonlinear homogeneous diffusions u^m, then depending on $m + k$ there are 3 regimes: fair-competition, diffusion-dominated and aggregation-dominated. Althought he was only working in 1D, it is clear from these particular situations that one can expect general results about long time asymptotics in each of the 3 cases. In fact, the classical Keller-Segel model for chemotaxis lies in the fair competition regime in which the homogeneity of both aggregation and diffusion terms is the same. For the diffusion-dominated case, existence and uniqueness of stationary solutions and its asymptotic stability was also treated. Where as for the aggregation-dominated case blow-up for certain initial data was obtained. This line of research will surely receive more attention in the next years to classify all possible behaviors. Interesting enough the use of gradient-flow techniques for this problem was crucial.

These results connect well with the question of the classical Keller-Segel problem for chemotaxis. This equation has an interaction potential given by the Newtonian potential and linear diffusion in two dimensions. The limiting singularity that Bertozzi, Laurent, and Rosado could allow in their theory is exactly the Newtonian potential in each dimension. This case corresponds to fair-competition with the denomination of the previous paragraph. Although this problem has been throughly studied during the last 30 years, there are still many open problems as already discussed above. Corrias's talk discussed issues related to the full parabolic Keller-Segel system and how to extend free-energy estimates to arrive to the same critical value as for the elliptic-parabolic system. Biler's talk showed the structure of self-similar solutions for the super critical cases.

Apart from methods involving Wasserstein like metrics, the talk of Daskalopoulos on ancient solutions of Ricci flow both presented recent progress and posed open problems, while the talk of Kinderlehrer raised many very interesting open questions about the use of entropies and about the nature of "entropy principles" in general.

A recent development that was very much in evidence at this meeting was the exciting progress that is being made in applying entropy methods to various biological models to discuss a wide range of phenomena, such a flocking, chemotaxis, trait evolution, aggregation and fragmentation, etc. The impact is not only on biology: applications of the methodology to pedestrian traffic and to economics were also made in talks at the meeting.

3 Scientific Progress Made

Considerable scientific progress was made at the meeting. One of the best ways to put this in evidence, given the time it takes for scientific progress to emerge in concrete form, is to look first at the progress that has emerged from the previous B.I.R.S. meeting on this subject in 2006.

One research program that took shape directly at that meeting was to develop an understanding of the fast diffusion equation for small values of the exponent m, so that if one write the equation in gradient flow form, the functional is not displacement convex, and also for *really small* values of m for which the corresponding Barenblatt profiles are no longer integrable. The program has been carried out with great success. Doing so involved the development of a number of new functional analytic tools. Reports on this work were made by Bonforte and McCann. This progress will lead to further progress. some of the tools used, like expansions on appropriately linearized equations, appear to be an emerging and rich problem, even for the simplest equations, like the fast-diffusion equation.

As noted in the report on the previous meeting, it sparked considerable interest in fourth order equations. That has lead to a number of advances. The work of Matthes, McCann and Savare on fourth order equations has already been mentioned, and at this meeting Ulusoy talked about joint work with Carlen on the thin film equation.

A good insight on various simplified models of chemotaxis and population dynamics has been achieved using entropy methods, and this is due in part to discussion taking place at the previous B.I.R.S. meeting.

As for the current meeting, it has once again provided a very valuable opportunity to get an snapshot of the status of the research in the field, and then, more importantly, to foster the interaction between different groups of mathematicians interested in nonlinear diffusion equations. As before, among these different groups we can mention mathematicians specialized in partial differential equations, geometric analysis, and the calculus of variations, however, in terms of of applications, there was this time an even wider range, with much more biology – indeed, there was more biology than physics, and this was certainly useful, as extensive familiarity with the mathematical problems of biology is less wide-spread in this community that has extensive familiarity with the mathematical problems of physics.

As anticipated, new perspectives were discovered, research collaborations fostered, and directions for future investigations determined. This workshop has certainly set the stage for future developments, new collaborations, and cross-pollination between different communities. Certainly, the impact of this event will be measured by relevant publications in the years to come.

Expectations set up during the application for the Banff conference were surpassed by the information given during the talks at the conference and the objectives of the proposal were fully met.

4 Outcome of the Meeting

4.1 Some observations

Four years after the workshop *Nonlinear diffusions: entropies, asymptotic behavior and applications* (15-20 April 2006), the current workshop was very appropriate. Significant progresses have been done by some of the participants and several techniques have reached a level of maturity that allows them to be considered as classical tools. A huge field of applications is now open, as well as a second wave of investigation on many remarkable aspects which have been noted over the past ten years but remain unexplained. We can for instance quote the issue of the duality through evolution aspects in Keller-Segel models, which allows to associate inequalities of interpolation of Gagliardo-Nirenberg type with inequalities involving mean field coupling like Hardy-Littlewood-Sobolev inequalities. Another interesting problem is to classify all possible long-time behaviors for the cases of homogeneous interaction kernels and diffusions. This will set up the strategy to cope with more general diffusive aggregation problems.

4.2 Abstracts of the talks

Some participants have not been able to attend the workshop for personal reasons or due to air traffic problems. They are mentioned with a $^{(*)}$ in the list of talks below. These abstracts can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Banff10w5054/program/

(login: Birs and password: 10w5054).

Slides of the presentations have also been collected and are available at

http://www.ceremade.dauphine.fr/∼dolbeaul/Banff10w5054/program/

(login: Birs and password: 10w5054).

AGUEH, MARTIAL. *Finsler structure in the p-Wasserstein space and applications to PDEs*
It is known from the work of F. Otto [*Comm. Partial Diff. Eq. 2001*], that the space of probability measures equipped with the quadratic Wasserstein distance, i.e., the 2-Wasserstein space, can be viewed as a "Riemannian manifold". Here we show that when the quadratic cost function is in general replaced by a homogeneous cost of degree $p > 1$, then the corresponding space of probability measures, i.e., the p-Wasserstein space, can be endowed with a Finsler metric whose induced distance function is the p-Wasserstein distance. Using this Finsler structure of the p-Wasserstein space, we give definitions of the differential and gradient of functionals defined on this space, and then of gradient flows in this space. In particular we show in this framework that the parabolic q-Laplacian equation is a gradient flow in the p-Wasserstein space, where $p = q/(q-1)$. When $p = 2$, we recover the Riemannian structure introduced by Otto, which confirms that the 2-Wasserstein space is a Riemann-Finsler manifold. Our approach is confined to a smooth situation where probability measures are absolutely continuous with respect to the Lebesgue measure on R^n, and they have smooth and strictly positive densities.

ANGENENT, SIGURD $^{(*)}$. *Contrasting Stochastic and non Stochastic Models for Cell Polarization*
We present the simplest possible model for cell polarization and show that in this model polarization of the cell membrane is cannot be explained by the continuum description of the model, and that persistent polarization, when it occurs, must be considered a stochastic effect. (joint work with Altschuler, Wang, and Wu).

BAKRY, DOMINIQUE $^{(*)}$. *Weighted Nash Inequalities*
In this talk, we shall present some families of weighted Nash type inequalities, which lead to non uniform upper bound to heat kernels. This may produce estimates on the Hilbert-Schmidt norm for Markov semigroups. We shall show on some 1-dimensional model examples that those inequalities are quite easy to obtain, even in the case where no ultracontractive or even hypercontractive bounds may be expected.

BILER, PIOTR. *Keller–Segel systems at $M = 8\,\pi$*
We consider models for chemotaxis

$$n_t = \Delta n - \nabla \cdot (n\,\nabla c)\,,$$

$$\tau\, c_t = \Delta c + n\,,$$

for $x \in \mathbb{R}^2$, $t > 0$, and for each $\tau \geq 0$.

As it is well known for the parabolic-elliptic system ($\tau = 0$), there is infinite number of locally stable steady states of critical mass $M = 8\pi$. On the other hand, initially not dispersed data lead to solutions that concentrate (or, in other words, blow up) in the infinite time: $n(\cdot\,, t) \rightharpoonup 8\,\pi\,\delta$ as $t \to \infty$.

The picture is much more complicated for the doubly parabolic Keller–Segel model ($\tau > 0$) since for each $\tau > \tau_*$ (with some $\tau_* > 0$), besides those steady states, there is a self-similar solution (n, c) of the form

$$\left(\frac{1}{t} u\left(\frac{x}{\sqrt{t}}\right),\, v\left(\frac{x}{\sqrt{t}}\right)\right)$$

also of mass $M = 8\,\pi$, which decays exponentially for $|x| \to\infty$. For this self-similar solution whole mass diffuses to infinity as $t \to \infty$.

BLANCHET, ADRIEN. *Functional inequalities, thick tails and asymptotics for the critical mass Patlak-Keller-Segel model*
We investigate the long time behaviour of the critical mass Patlak-Keller-Segel equation. This equation has a one parameter family of steady-state solutions $\rho_{\infty,\lambda}$, $\lambda > 0$, with thick tails whose second moment is not bounded. We show that these steady state solutions are stable, and find basins of attraction for them using an entropy functional $\mathcal{H}$ coming from the critical fast diffusion equation in $\mathbb{R}^2$. We construct solutions of Patlak-Keller-Segel equation satisfying an entropy-entropy dissipation inequality for $\mathcal{H}$. While the entropy dissipation for $\mathcal{H}$ is strictly positive, it turns out to be a difference of two terms, neither of which need to be small when the dissipation is small. We introduce a strategy of *controlled concentration* to deal with this issue, and then use the regularity obtained from the entropy-entropy dissipation inequality to prove the existence of basins of attraction for each stationary state composed by certain initial data converging towards

$\rho_{\infty,\lambda}$. In the present talk, we do not provide any estimate of the rate of convergence, but we discuss how this would result from a stability result for a certain sharp Gagliardo-Nirenberg-Sobolev inequality.

BONFORTE MATTEO. *Asymptotics of the Fast Diffusion Equation via Entropy methods*
We consider non-negative solutions of the fast diffusion equation $u_t = \Delta u^m$ with $m \in (0,1)$, in the Euclidean space $\mathbb{R}^d$, $d \geq 3$, and study the asymptotic behavior of a natural class of solutions, in the limit corresponding to $t \to \infty$ for $m \geq m_c = (d-2)/d$, or as t approaches the extinction time when $m < m_c$. For a class of initial data we prove that the solution converges with a polynomial rate to a self-similar solution, for t large enough if $m \geq m_c$, or close enough to the extinction time if $m < m_c$. Such results are new in the range $m \leq m_c$ where previous approaches fail. In the range $m_c < m < 1$ we improve on known results.

The precise value for the exponent $m = (d-4)/(d-2)$, $d \geq 3$, requires quite different functional analytic methods, due in particular to the absence of a spectral gap for the linearized generator.

CALVEZ, VINCENT. *Gradient flow interpretation for the one-dimensional Keller-Segel equation.*
Joint work with J.A. Carrillo (ICREA, UAB, Barcelona, Spain).

We wish to present a brief overview of recent developments in the analysis of the generalized Keller-Segel equation in dimension one:

$$\partial_t \rho(t,x) = \partial_{xx} \rho^m(t,x) + \chi \partial_x \left(\rho(t,x) \partial_x \left(\frac{|x|^k}{k} * \rho(t,x) \right) \right), \quad t > 0, \quad x \in \mathbb{R},$$

for the following set of exponents: $m \geq 1$ and $k \in (-1,1)$. Here $\rho(t,x)$ denotes a density of cells which diffuse and attract each other through a mean field potential. This model has raised a lot of interest in the field of mathematical biology since it generally exhibits a dichotomy between global existence (dispersion of the cells) and finite time blow-up (aggregation of the cells).

This equation possesses the structure of a gradient flow for the Wasserstein metric on the space of probability densities. Interestingly enough the energy functional $\mathcal{F}$ is the sum of two opposite contributions, being respectively displacement-convex and concave, and homogeneous. In particular we obtain some blow-up criterion as a direct consequence of the homogeneity property in the case $m + k < 1$ (*attraction-dominating competition*):

$$\mathcal{F}[\rho_0] < C \frac{(m-1+k)}{k} \left(\int_{\mathbb{R}} |x|^2 \rho_0(x)\, dx \right)^{(1-m)/2}.$$

On the other hand we investigate the long-time asymptotics in the case $m + k = 1$ (*fair competition*). We are able to prove the following inequalities along the gradient flow in appropriately rescaled variables:

$$\frac{1}{2} \frac{d}{dt} W(\rho(t), \mu)^2 + W(\rho(t), \mu)^2 \leq 0, \quad (\mathbf{m = 1}),$$

$$\frac{1}{2} \frac{d}{dt} W(\rho(t), \mu)^2 + \frac{m+1}{2} W(\rho(t), \mu)^2 \leq (m-1) \left(\mathcal{F}[\rho(t)] - \mathcal{F}[\mu] \right), \quad (\mathbf{m > 1}),$$

where μ denotes the minimizer of the energy functional (supplemented with a quadratic confinement potential due to rescaling). We deduce from the former a uniform rate of convergence towards the minimizer in the case $(m = 1, k = 0)$. These inequalities are strongly related to a convex-like property of the energy functional.

CANIZO, JOSÉ ALFREDO. *Entropy-entropy dissipation inequalities for fragmentation-drift equations*
The growth-fragmentation equation models a group of cells which grow (or age) at a certain rate, and divide into two or more pieces. After a suitable rescaling, the shape of the distribution function of the population converges to a universal profile. By means of entropy-entropy dissipation inequalities for some unbounded fragmentation coefficients, we show that the speed of this convergence is exponential. This kind of inequalities may be used for other linear equations for which the General Relative Entropy principle is valid.

CHUGUNOVA, MARINA. *Entropy-energy analysis of coating flows*
The equation

$$u_t + [u^n(u_{xxx} + \gamma^2 u_x - \sin(x))]_x = 0$$

with periodic boundary conditions is a model of the evolution of a thin liquid film on the outer surface of a horizontal cylinder in the presence of gravity field. We use entropy-energy methods to study different properties of generalized weak solutions of this equation. For example: finite speed of the compact support propagation for $n \in (1,3)$ is proved by application of local α-entropy estimates.

Joint work with A. Burchard, M. Pugh, B. Stephens, and R. Taranets

CORRIAS, LUCILLA. *The role of the free energy and its dissipation in the mathematical analysis of the parabolic-parabolic Keller-Segel system.*
It is well known that the classical Keller-Segel system modeling chemotaxis is endowed with a free energy functional decreasing along the trajectories of the solutions. In this talk, we shall show how to handle that energy with the help of several sharp functional inequalities, in order to obtain both global existence results and density concentration phenomena for the fully parabolic Keller-Segel system on the whole space. The case of the high space dimension is also considered. (Joint works with V. Calvez and A. Ebde).

DASKALOPOULOS, PANAGIOTA. *Ancient solutions to the Ricci flow on Surfaces*
We provide a classification for ancient solutions to the Ricci flow on S^2. We show that they are either the contracting spheres or the King-Rosenau solutions. We also prove a similar result for ancient convex embedded solutions to the curve shortening flow. This is joint work with Richard Hamilton and Natasa Sesum.

DESVILLETTES, LAURENT. *Entropy methods and cross diffusions*
Some models in population dynamics include so-called cross diffusions terms. Those terms model the effect of the presence of one type of individuals on the diffusion rate of other individuals. We shall present in this talk results about the modeling and the mathematical analysis of systems including cross diffusion: those results share a common feature: the use of Lyapounov-like functionals (entropies).

DI FRANCESCO, MARCO. *On the Hughes' model for pedestrian flow: The one-dimensional case*
In this talk we investigate the mathematical theory of Hughes' model for the flow of pedestrians (cf. Hughes 2002), consisting of a nonlinear conservation law for the density of pedestrians coupled with an eikonal equation for a potential modelling the common sense of the task. We first consider an approximation of the original model in which the eikonal equation is replaced by an elliptic approximation.

For such an approximated system we prove existence and uniqueness of entropy solutions (in one space dimension) in the sense of Kruzkov, in which the boundary conditions are posed following the approach of Bardos et al.. We use BV estimates on the density and stability estimates on the potential in order to prove uniqueness.

Furthermore, we analyse the evolution of characteristics for the original Hughes' model in one space dimension and study the behaviour of simple solutions, in order to reproduce interesting phenomena related to the formation of shocks and rarefaction waves. The characteristic calculus is supported by numerical simulations.

FELLNER, KLEMENS. *On non-local interaction equations: Stbility of Stationary States and Limiting Behaviour*
We study non-local evolution equations for a density of individuals, which interact through a given symmetric potential. Such models appear in many applications such as swarming and flocking, opinion formation, inelastic materials, ... In particular, we are interested in interaction potentials, which behave locally repulsive, but aggregating over large scales. A particular example for such potentials was recently given in models of the alignment of the directions of filaments in the cytoskeleton.

We present results on the structure and stability of steady states. We shall show that stable stationary states of regular interaction potentials generically consist of sums of Dirac masses. However the amount of Diracs depends delicately on the interplay between local repulsion and aggregation. In particular we shall see that singular repulsive interaction potentials introduce diffusive effects in the sense that stationary state are rendered continuously.

FIGALLI, ALESSIO. *A new transportation distance between non-negative measures*
Given a bounded domain $\mathcal{O}$, it is by now well-known that that the gradient flow of the entropy functional $\int_{\mathcal{O}} [\rho \log(\rho) - \rho]\,dx$ with respect to the Wasserstein distance produces a solution to the heat equation with Neumann boundary conditions. Recently, in collaboration with Nicola Gigli we introduced a new transportation distance between non-negative measures inside a domain $\mathcal{O}$. This distance enjoys many nice properties, for instance it makes the space of non-negative measures inside $\mathcal{O}$ a geodesic space, without any convexity

assumption on $\mathcal{O}$. Moreover, the gradient flow of the entropy functional $\int_{\mathcal{O}}[\rho\log(\rho)-\rho]\,dx$ w.r.t. this distance coincides with the heat equation subject to the Dirichlet boundary condition equal to 1. The aim of this talk will be to briefly review the classical theory, and then to introduce this new distance and its main properties.

GONZALEZ, MARIA DEL MAR. *A free-boundary model in price formation*
We study a model, due to J.M. Lasry and P.L. Lions, describing the evolution of a scalar price which is realized as a free boundary in a 1D diffusion equation with dynamically evolving, non-standard sources. We establish global existence and uniqueness. This is joint work with L. Chayes, M. Gualdani and I. Kim.

KINDERLEHRER, DAVID. *Coarsening in cellular systems*
Mesoscale experiment and simulation permit harvesting information about both geometric features and texture in material microstructures. The grain boundary character distribution (GBCD) is an empirical distribution of the relative length (in 2D) or area (in 3D) of interface with a given lattice misorientation and grain boundary normal. During the growth process, an initially random texture distribution reaches a steady state that is strongly correlated to the interfacial energy density. In simulation, we found that if the given energy depends only on lattice misorientation, then the steady state GBCD and the energy are related by a Boltzmann distribution. This is among the simplest non-random distributions, corresponding to independent trials with respect to the energy. Why does such a simple distribution arise from such a complex system?

We outline an entropy based theory which suggests that the evolution of the GBCD satisfies a Fokker-Planck Equation. Coarsening in polycrystalline systems is a complicated process involving details of material structures, chemistry, arrangement of grains in the configuration, and environment. In this context, we consider just two global features: cell growth according to a local evolution law of curvature driven growth and space filling constraints. Space filling requirements are managed by critical events, rearrangements of the network involving deletion of small contracting cells and facets. The interaction between the evolution law and the constraints is governed primarily by the Herring Condition, the boundary condition associated with the equation of curvature driven growth. It determines a dissipation relation. To assist in the derivation, a simpler system is introduced which is driven by the boundary conditions and reflects the dissipation relation of the grain growth system. It resembles an ensemble of inertia-free spring-mass-dashpots. For this simpler coarsening network, we learn how entropic or diffusive behavior at the large scale emerges from a dissipation relation at the scale of local evolution.

Joint work with K. Barmak, E. Eggeling, M. Emelianenko, Y. Epshteyn, R. Sharp, and S. Ta'asan.

LAURENCOT, PHILIPPE. *Large time behaviour for quasilinear a degenerate diffusion equation with gradient source term*
The qualitative behaviour of nonnegative solutions to the Cauchy-Dirichlet problem for the equation $\partial_t u - \Delta_p u = |\nabla u|^q$ in a bounded domain depends on the relative strength of the diffusion and the source terms. According to the values of p and q, multiple steady states may exist or convergence to a "friendly giant" may take place or finite time blowup may occur (joint works with G. Barles, C. Stinner, and M. Winkler).

LAURENT THOMAS. *Mechanism of formation of Dirac masses in the solutions of the aggregation equation*
The aggregation equation is a continuum model for interacting particle systems with attractive/repulsive pairwise interaction potential. It arises in a number of models for biological aggregation, but also in materials science and granular media. As a mathematics problem, the aggregation equation is a gradient flow of the interaction energy with respect to the Wasserstein distance. The main phenomenon of interest is that, even with smooth initial data, the solutions can concentrate mass in finite time (i.e. a delta Dirac appears in the solution in finite time). We study the mechanism of formation of these delta Dirac masses.

LEE, PAUL WOON YIN. *Generalized Ricci Curvature Bounds for Three Dimensional Contact Subriemannian Manifolds*
Measure contraction property (MCP) is one of the possible generalizations of Ricci curvature bound to more general metric measure spaces. However, the definition of MCP is not computable in general. In this talk, I'll discuss computable sufficient conditions for a three dimensional contact subriemannian manifold to satisfy such property. This is a joint work with Andrei Agrachev.

LISINI, STEFANO. *Measure solutions of sub-linear diffusion equations with drift*

We present some recent results, obtained in collaboration with Giuseppe Savaré, Giuseppe Toscani and Simona Fornaro, about non-negative measure valued solutions of the Cauchy problem for a one-dimensional drift-diffusion equation of the form

$$\partial_t \rho - \partial_x \big(\partial_x(\beta(\rho)) + V'\rho\big) = 0, \quad \text{in } (0,+\infty)\times\mathbb{R}; \quad \rho(0,\cdot) = \rho_0 \quad \text{in } \mathbb{R}. \tag{3}$$

The diffusion function β is increasing, smooth, and satisfies

$$\lim_{r\to+\infty} \beta(r) < +\infty, \tag{4}$$

whereas $V : \mathbb{R} \to \mathbb{R}$ is a λ-convex driving potential. The initial datum ρ_0 is a non-negative Borel measure with finite total mass $\mathfrak{m}_0 := \rho_0(\mathbb{R})$ and finite quadratic momentum.

By the assumption (4), in the regions where ρ is large the diffusion term in (3) could be negligible and the drift term in (3) could be predominant. Based on the previous observation, when the initial measure ρ_0 has a not bounded density or has a singular part, the presence of a singular part in the solution ρ of (3) has to be taken into account.

We give a suitable notion of measure valued solution of problem (3) and we show existence and uniqueness. Moreover we show a characterization of the evolution of the singular part of the solution.

Finally we show that the large time behavior of the solutions of (3) depends on a critical mass $\mathfrak{m}_c$ ($\mathfrak{m}_c$ depending on β and V). If the initial mass $\mathfrak{m}_0 \le \mathfrak{m}_c$, then there is a non singular steady state, while in the case $\mathfrak{m}_0 > \mathfrak{m}_c$ the steady state has a singular part.

The proof of existence and uniqueness is based on the interpretation of the solutions of problem (3) as trajectories of the gradient flow of the energy functional

$$\mathcal{E}(\rho) := \int_{\mathbb{R}} E(u(x))\,\mathrm{d}x + \int_{\mathbb{R}} V(x)\,\mathrm{d}\rho(x) \quad \text{where} \quad \rho = u\mathrm{d}x + \rho^{\perp},$$

(here $\beta'(r) = rE''(r)$) with respect to the Wasserstein distance between non negative measures with mass $\mathfrak{m}_0$.

MATTHES, DANIEL. *Gradient flow of the Diriclet functional*

On a formal level, the Cahn-Hillard equation $\partial_t u = -\mathrm{div}\big(\mathbf{m}(u)\nabla\Delta u\big) + \Delta P(u)$ is readily seen to constitute a gradient flow of a perturbed Diriclet energy functional. The properties of the associated metric — which is a Wasserstein-like distance, but with the *non-linear mobility* function $\mathbf{m}$ — have been investigated recently by Dolbeault, Nazareth and Savaré. In this talk, we show how their results can be used to make the formal variational structure of the Cahn-Hilliard equation rigorous. Our basic assumption is that $\mathbf{m}$ is concave and satisfies some growth condition.

In application of the rigorous gradient flow structure, we derive sharp estimates on the intermediate asymptotics of weak solutions to the Hele-Shaw flow. The latter corresponds to the Cahn-Hilliard equation with *linear* mobility $\mathbf{m}$, when the associated metric is just the classical L^2-Wasserstein distance. The key idea is to use a very particular connection between the Diriclet energy, an entropy and the Wasserstein distance.

This is joint work with Stefano Lisini and Giuseppe Savaré (Pavia).

MCCANN, ROBERT. *Higher order asymptotics of fast diffusion in Euclidean space*

With Denzler and Koch, we quantify the speed of convergence and higher-order asymptotics of fast diffusion dynamics on $\mathbf{R}^n$ to the Barenblatt (self similar) solution. Degeneracies in the parabolicity of this equation are cured by re-expressing the dynamics on a manifold with a cylindrical end, called the cigar. The nonlinear evolution semigroup becomes differentiable in Hölder spaces on the cigar. The linearization of the dynamics is given by the Laplace-Beltrami operator plus a transport term (which can be suppressed by introducing appropriate weights into the function space norm), plus a finite-depth potential well with a universal profile. In the limiting case of the (linear) heat equation, the depth diverges, the number of eigenstates increases without bound, and the continuous spectrum recedes to infinity. We provide a detailed study of the linear and nonlinear problems in Hölder spaces on the cigar, including a sharp boundedness estimate for the semigroup, and use this as a tool to obtain sharp convergence results toward the Barenblatt solution (as Bonforte, Dolbeault, Grillo and Vazquez also did independently in a weaker metric), and higher order asymptotics. In finer convergence results (after modding out symmetries of the problem), a subtle interplay between convergence rates and tail behavior is revealed. The difficulties involved in choosing the right functional spaces in which to carry out the analysis can be interpreted as genuine features of the equation rather than mere annoying technicalities.

NAZARET, BRUNO. *Distances on probability measures induced by concave mobilities : geodesics via PDE*
In the framework of distances on probability measures introduced by Dolbeault, Nazaret and Savaré, we present some cases where the geodesics are obtained as solutions to a system of PDE on the density and the velocity of the intermediate configurations (joint work with F. Santambrogio).

RAOUL GAËL. *Selection dynamics for the evolution of traits in a population*
We consider a model describing the evolution of a large population of individuals with different biological ”traits”. The reproduction rate of each individual depends on its trait and on the competition for resources within the population. For a certain class of coefficients, an entropy exists, and can be used to study the large time behaviour of the population, and in particular the formation of Dirac masses, that corresponds to species formation.

ROSADO, JESUS. *A refined result of flocking for the Cucker-Smale model*
I will present and analyse the asymptotic behaviour of solutions of the continuous kinetic version of flocking by Cucker and Smale [2], which describes the collective behaviour of an ensemble of organisms, animals or devices. This kinetic version introduced by Ha and Tadmor is obtained from a particle model. The large-time behaviour of the distribution in phase space is subsequently studied by means of particle approximations and a stability property in distances between measures. A continuous analogue of the theorems of Cucker and Smale will be shown to hold for the solutions on the kinetic model. More precisely, the solutions concentrate exponentially fast their velocity to their mean while in space they will converge towards a translational flocking solution. The presentation is based on [1].

SAVARÉ, GIUSEPPE. *Mobility functions, transport distances, and nonlinear diffusion*
We present a new class of distances between nonnegative measures in the Euclidean spaces. They are modeled on the dynamical characterization of the Wasserstein distances proposed by Benamou-Brenier and provide a wide family interpolating between the Wasserstein and the homogeneous Sobolev distances.

From the point of view of optimal transport theory, these distances minimize a dynamical cost to move a given initial distribution of mass to a final configuration. An important difference with the classical setting in mass transport theory is that the cost not only depends on the velocity of the moving particles but also on the densities of the intermediate configurations with respect to a given reference measure.

New existence, stability, and contraction results for solutions of nonlinear diffusion equations in a convex bounded domain will be discussed. (In collaboration with J. Dolbeault, B. Nazaret, J. Carrillo, S. Lisini, D. Slepčev)

SCHMEISER, CHRISTIAN. *Hypocoercivity of a kinetic model for fast diffusion*
Convergence to equilibrium will be discussed for BGK kinetic models, which have fast diffusion equations as their macroscopic limits.

ŠEŠUM. NATAŠA. *Curvature flows*
We will discuss various curvature flows such as the harmonic mean curvature flow (HMCF), the mean curvature flow (MCF) and the Ricci flow (RF). We will discuss the long time behaviour of the HMCF in the case of star shaped surfaces and also weakly convex hypersurfaces in higher dimensions. We will also discuss the curvature conditions that guarantee the smooth existence of the MCF and the RF.

STAŃCZY, ROBERT. *Entropy methods for self-gravitating particles systems*
Systems describing the interaction of gravitationally attracting particles that obey some statistics are studied. The role of the entropy for these system is thoroughly analyzed both in the isothermal case and in the case with the energy fixed. Systems of a similar form and structure appear also in modelling the chemotaxis phenomena in biology.

STURM, KARL-THEODOR. *Monotone Approximation for the Wasserstein Diffusion*
We present a system of interacting Brownian particles on the real line for which the empirical distributions – in the scaling limit of large particle numbers – converge to the Wasserstein diffusion. The latter is a reversible

Markov process with continuous paths on the space of probability measures whose square field operator – which governs the short time behavior – is the squared norm of the Wasserstein gradient. We indicate also some extensions to higher dimensional spaces.

ULUSOY, SULEYMAN. *Long Time Behavior of Weak Solutions in Hele-Shaw Flow Problem*
We investigate the long-time behavior of weak solutions to the thin- film type equation $u_t = -(uu_{xxx})_x$. We employ a semidiscrete variational scheme to generate weak solutions as a gradient flow with respect to so called Wasserstein distance and we show that these weak solutions converge to the unique self-similar source type solution exponentially fast. This paper complements our results in Carlen, E. A., Ulusoy, S.: Asymptotic equipartition and long time behavior of solutions of a thin-film equation, J. Differential Equations., 241, pp. 279-292, 2007. This is a joint work with Eric A. Carlen.

WESTDICKENBERG, MICHAEL. *Optimal Transport for the System of Isentropic Euler Equations*
The isentropic Euler equations form a system of conservation laws modeling compressible fluid flows with constant thermodynamical entropy. Due to the occurrence of shock discontinuities, the total energy of the system is decreasing in time. We review the second order calculus on the Wasserstein space of probability measures and show how the isentropic Euler equations can be interpreted as a steepest descent equation in this framework. We introduce a variational time discretization based on a sequence of minimization problems, and show that this approximation converges to a suitably defined measure-valued solution of the conservation law. Finally, we present some preliminary results about the numerical implementation of our time discretization.